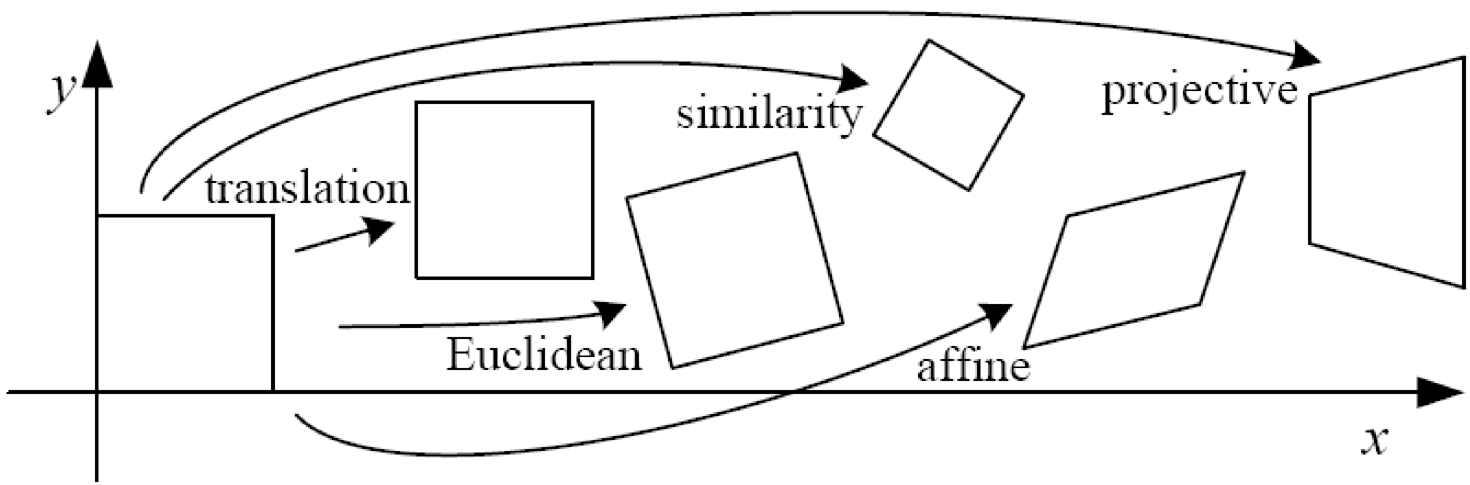
Geometric transformation



References

- http://szeliski.org/Book/
- http://www.cs.cornell.edu/courses/cs5670/2019sp/lectures/lectures.html
- http://www.cs.cmu.edu/~16385/

contents

- **2D->2D transformations**
- 3D->3D transformations
- 3D->2D transformations (3D projections)
  - Perspective projection
  - Orthographic projection

Objective

- Being able to do all of the below transformations with matrix manipulation:

translation

rotation

scale

shear

affine

projective

- **Why matrix manipulation?**

Objective

- Being able to do all of the below transformations with matrix manipulation:


translation


rotation


scale

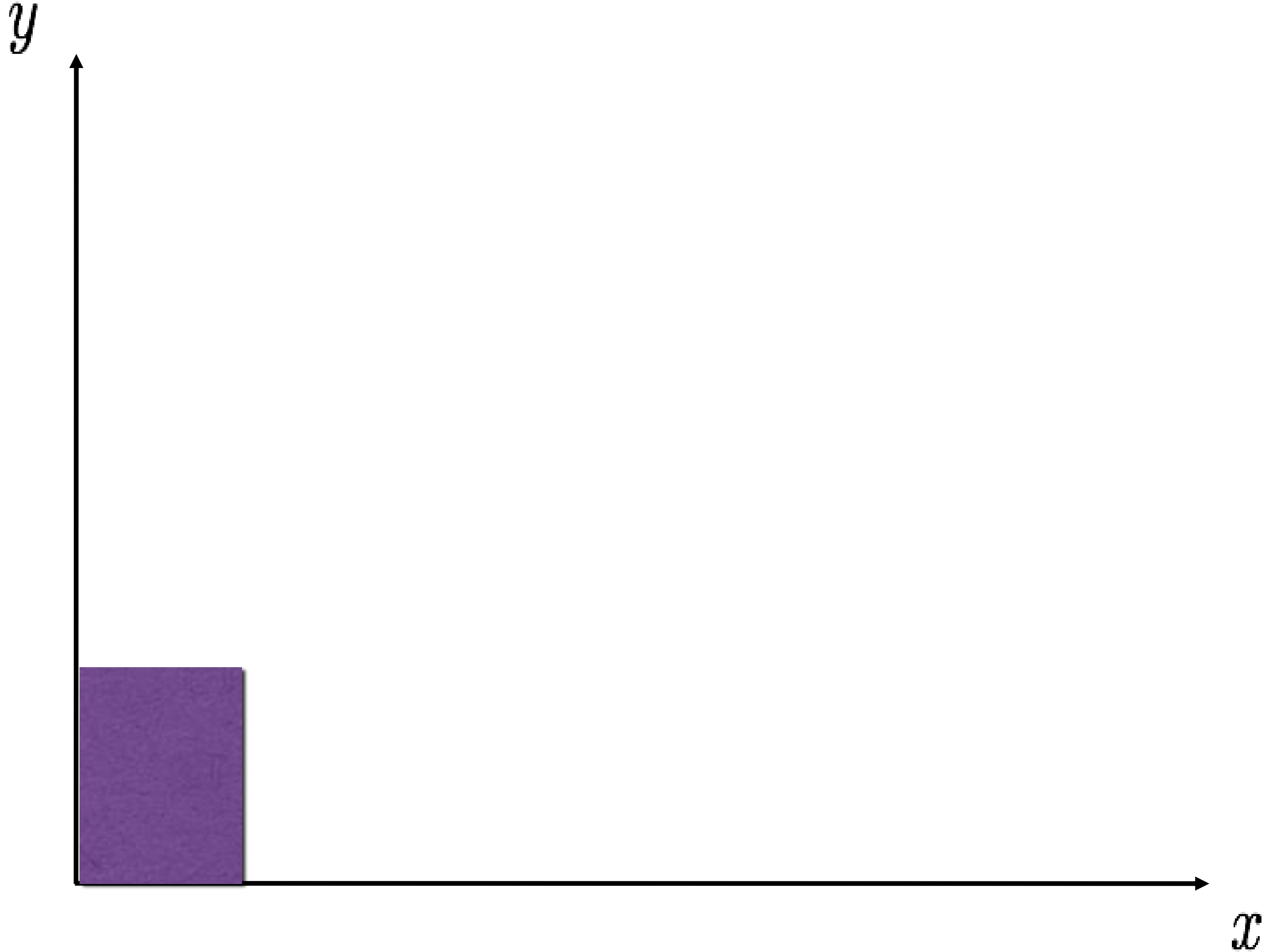

shear


affine


projective

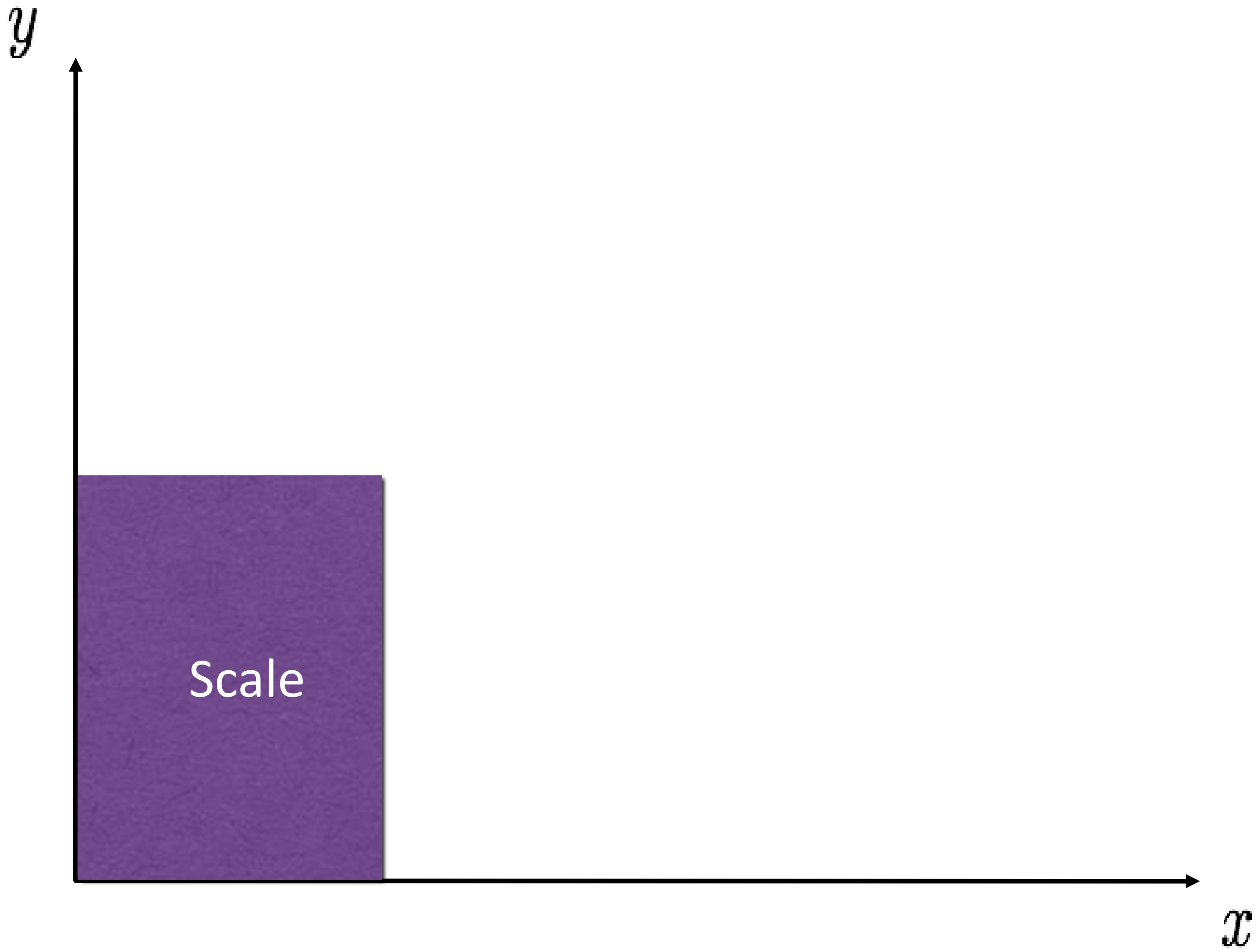
- **Why matrix manipulation?** Because then we can easily concatenate transformations (for example translation and then rotation).

2D planar transformations

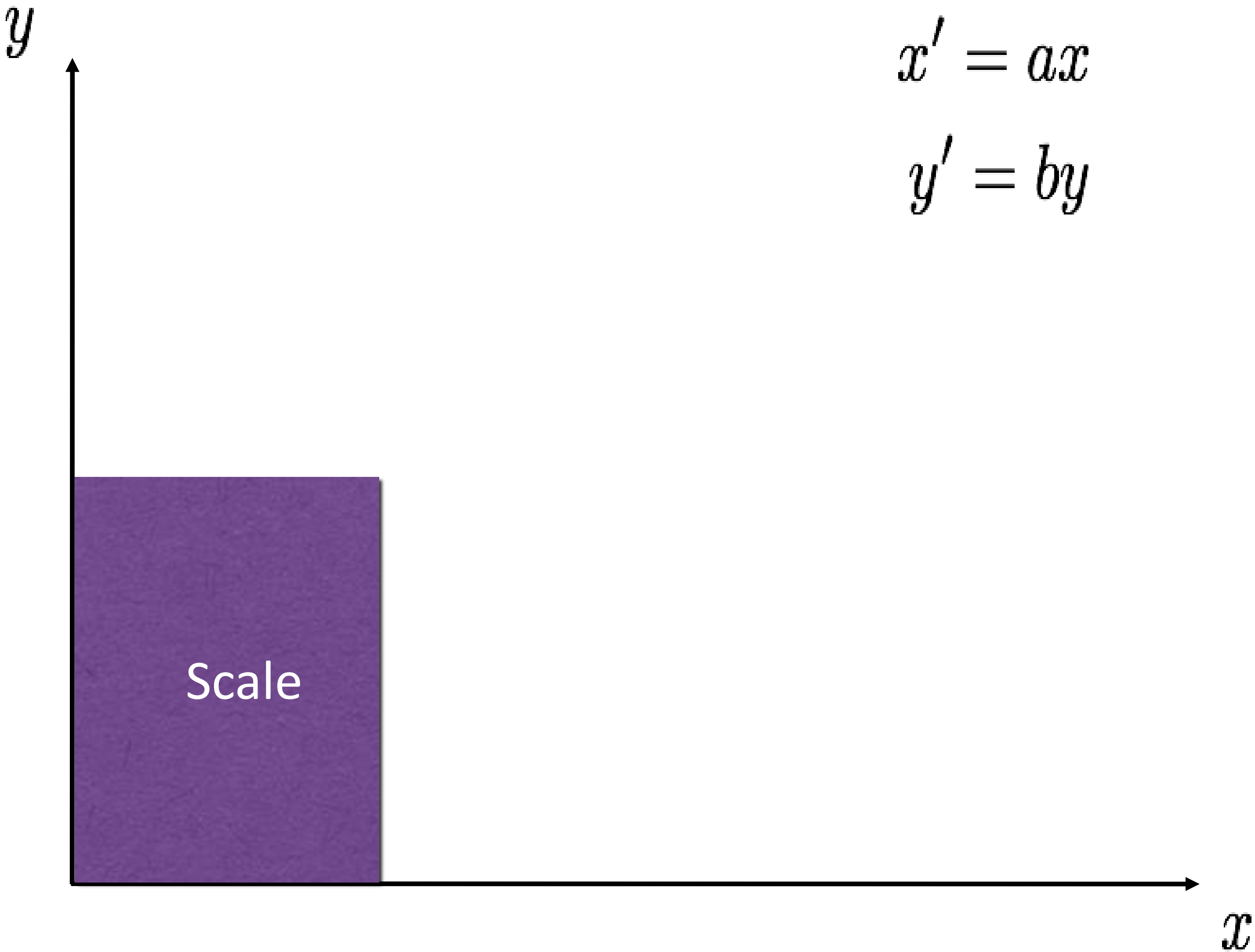


scale

- How?



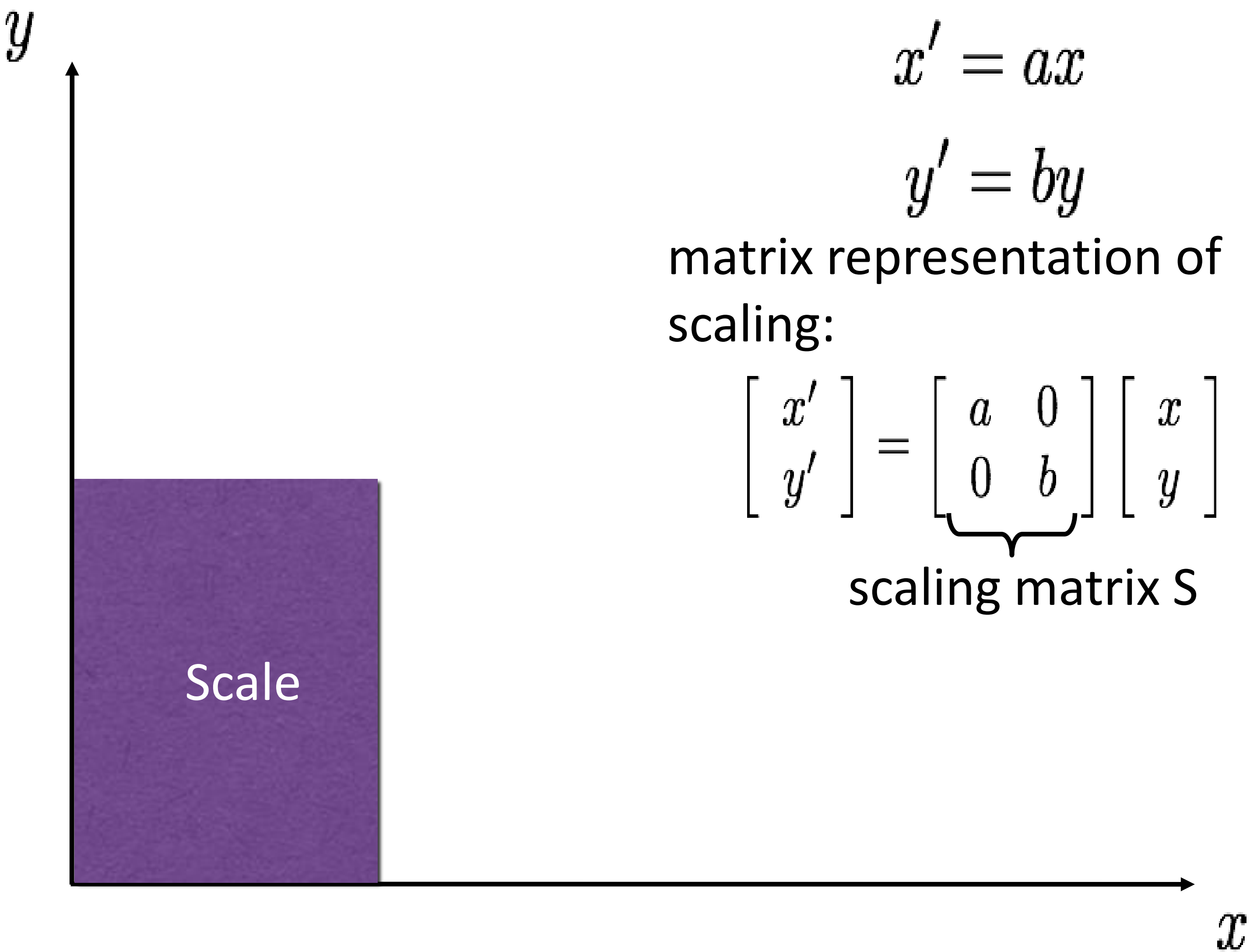
scale



$$x' = ax$$

$$y' = by$$

scale



$$x' = ax$$

$$y' = by$$

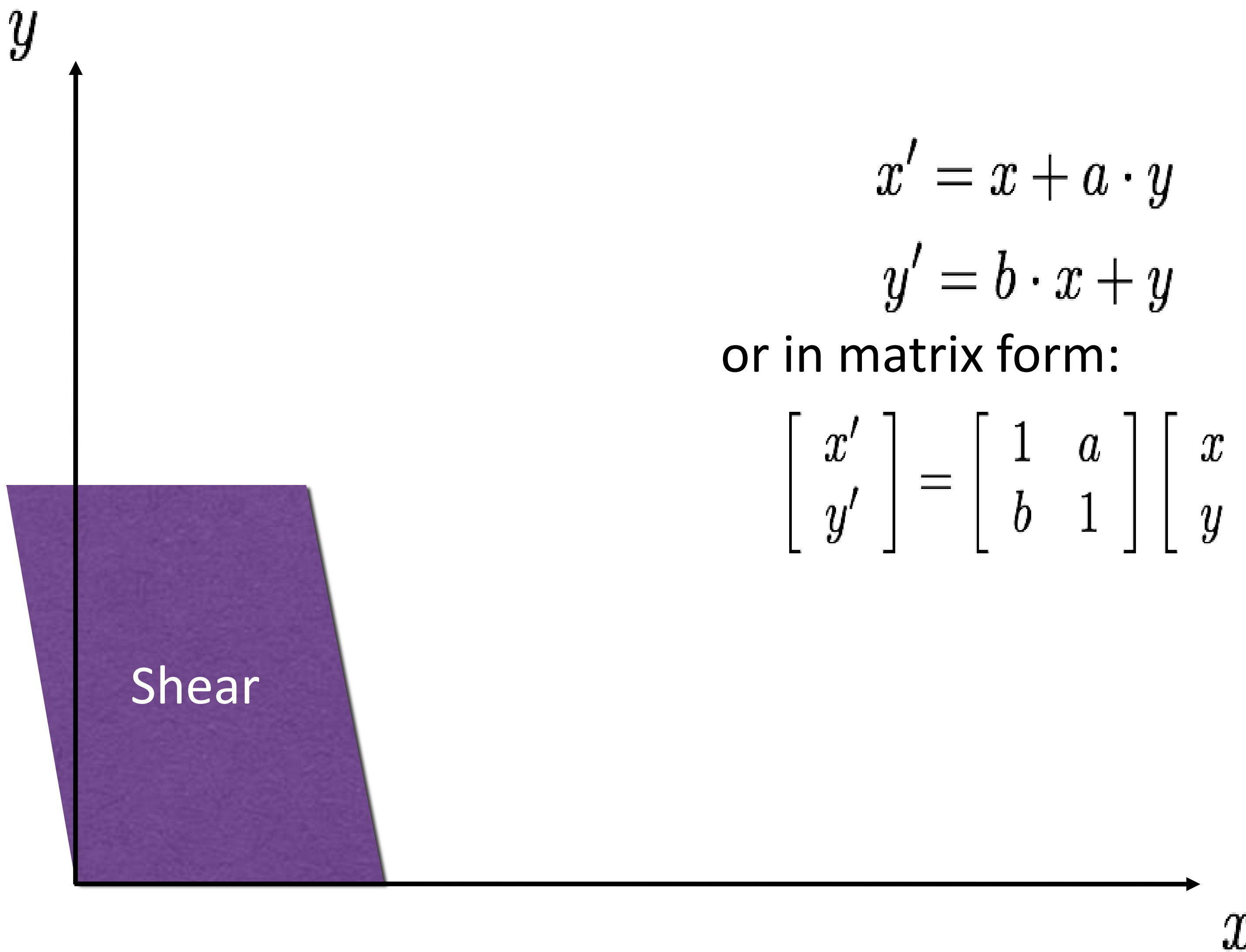
matrix representation of scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear

- How?

Shear



$$x' = x + a \cdot y$$

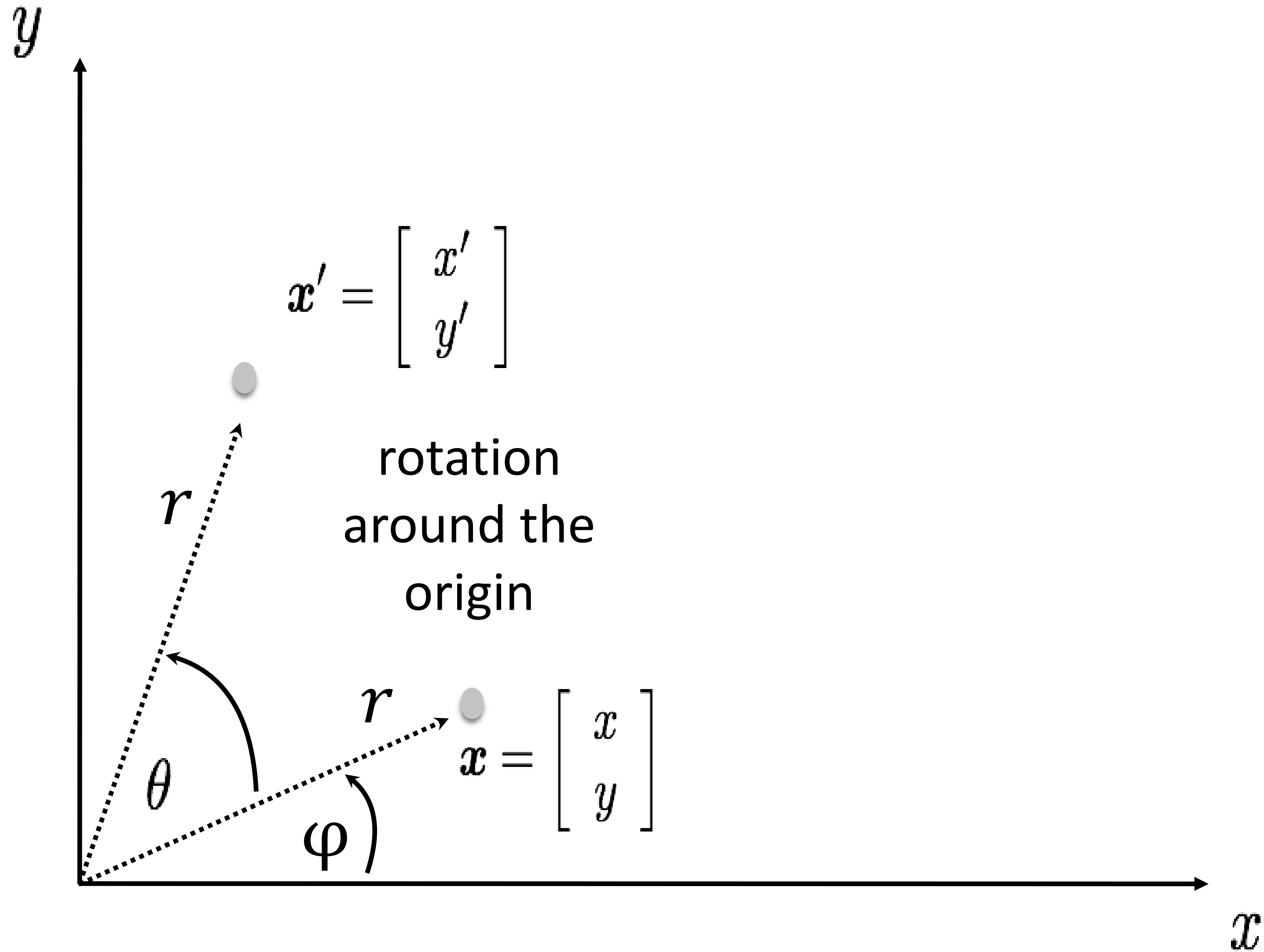
$$y' = b \cdot x + y$$

or in matrix form:

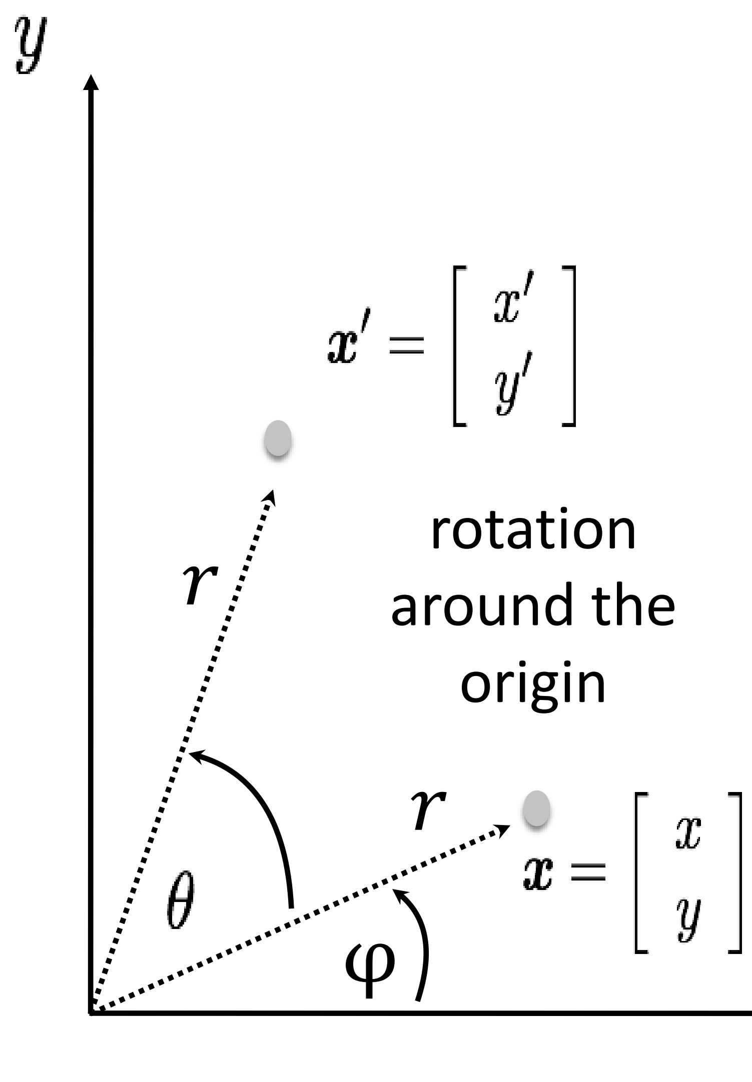
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation

- How?



Rotation



Polar coordinates…

$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trigonometric Identity…

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

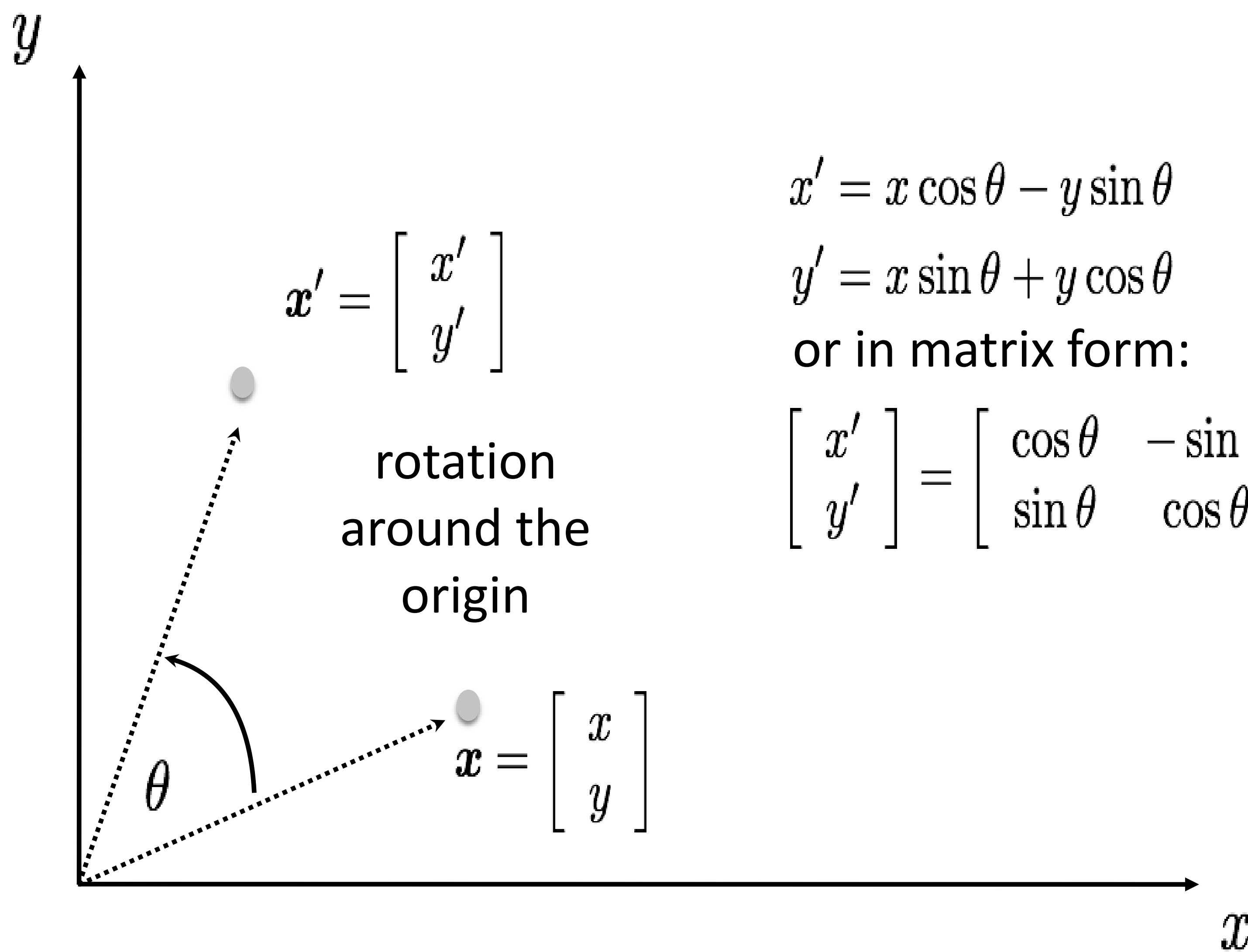
$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…

$x' = x \cos(\theta) - y \sin(\theta)$

$y' = x \sin(\theta) + y \cos(\theta)$

Rotation



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Important rotation matrix features

- $\det(R) = 1$
  - If $\det(R) = -1$ then this is a roto-reflection matrix

- $R^T = R^{-1} \leftrightarrow RR^T = R^T R = I \leftrightarrow$ **orthogonal matrix** $\leftrightarrow$ a square matrix whose **columns and rows are orthogonal unit vectors**.

Concatenation

- How do we do concatenation of two or more transformations?

Concatenation

- How do we do concatenation of two or more transformations?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and then } \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

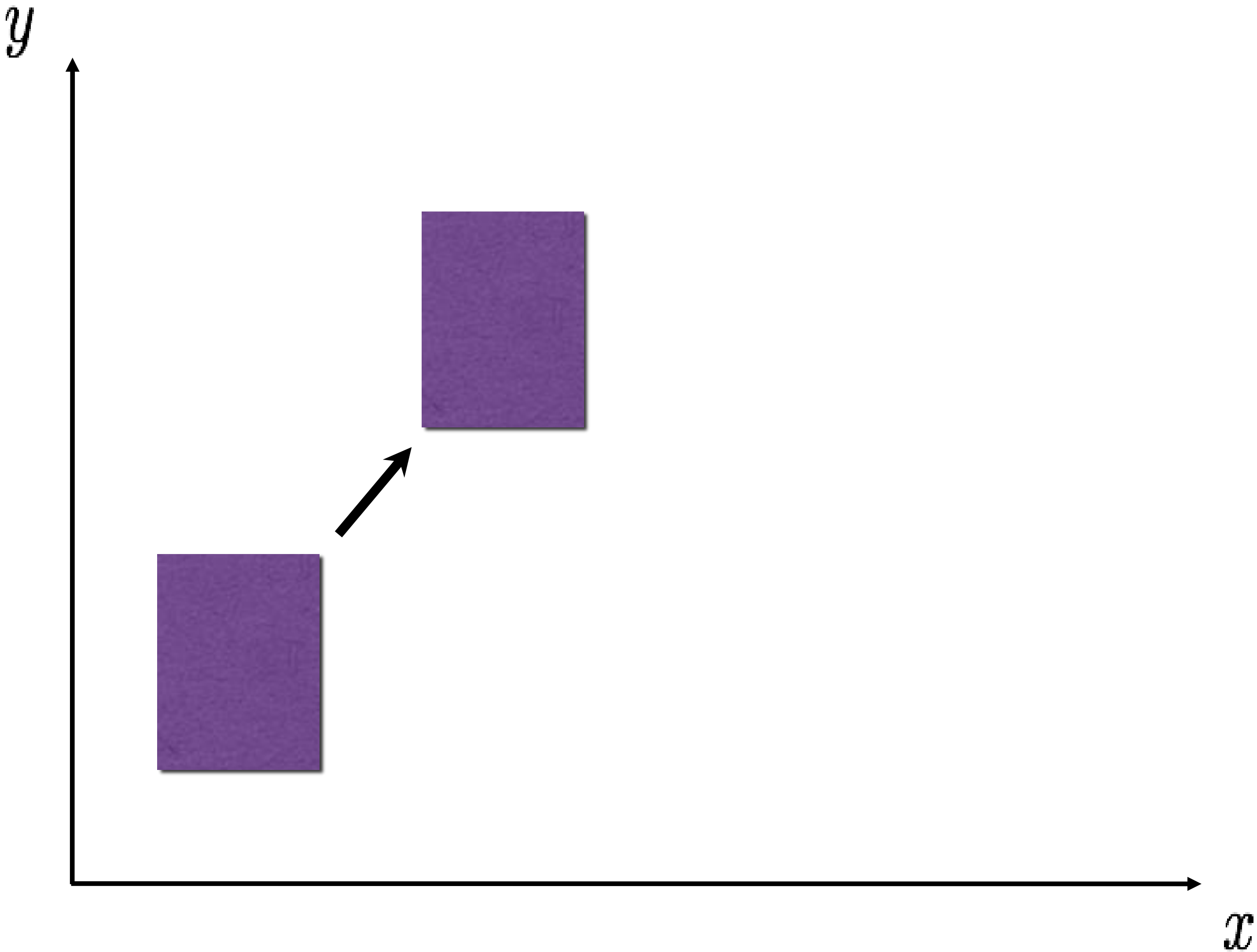
$$\mapsto$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} acos\theta & -bsin\theta \\ asin\theta & bcos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

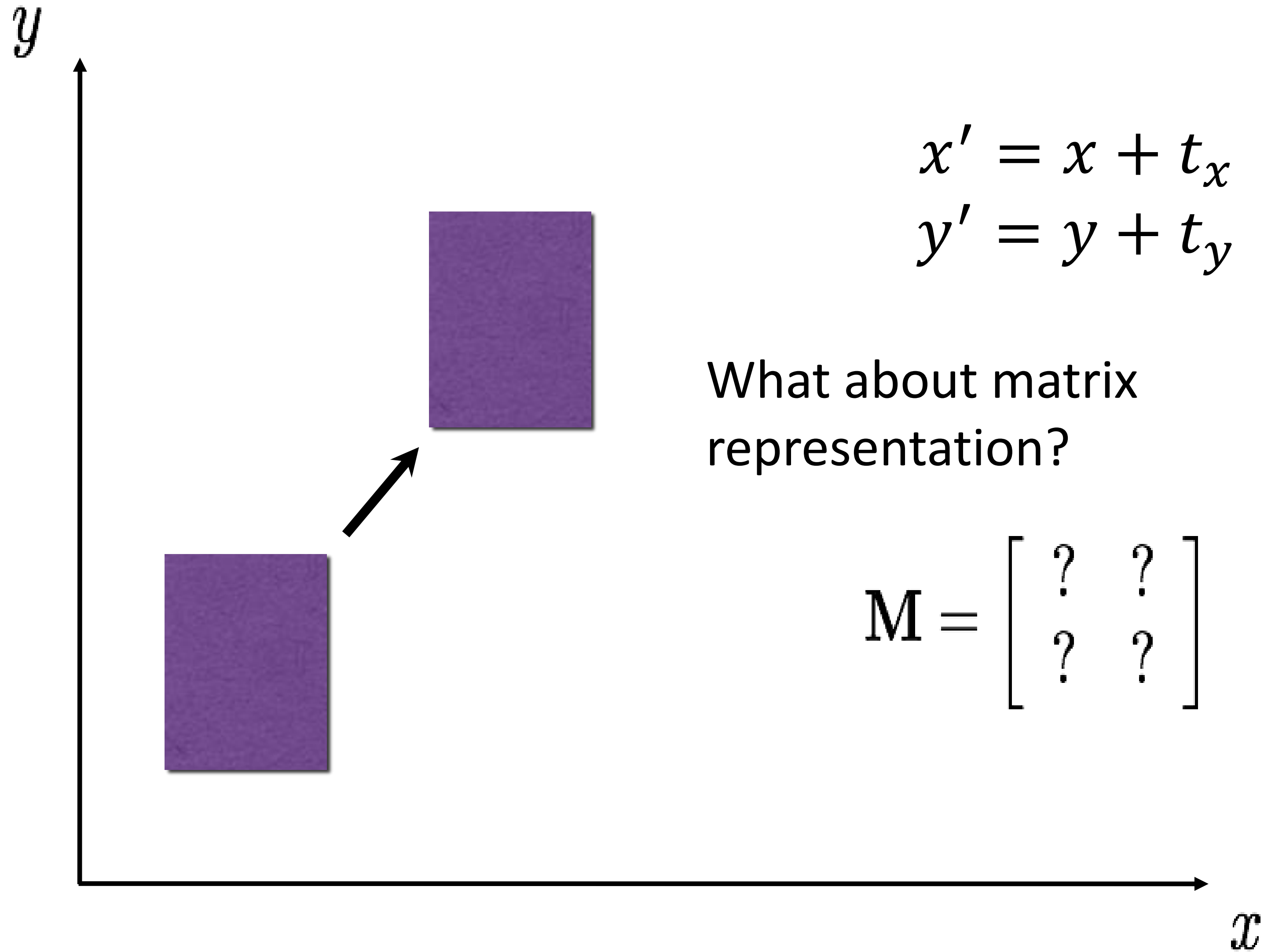
- Easy with matrix multiplication!

Translation

- How?



Translation

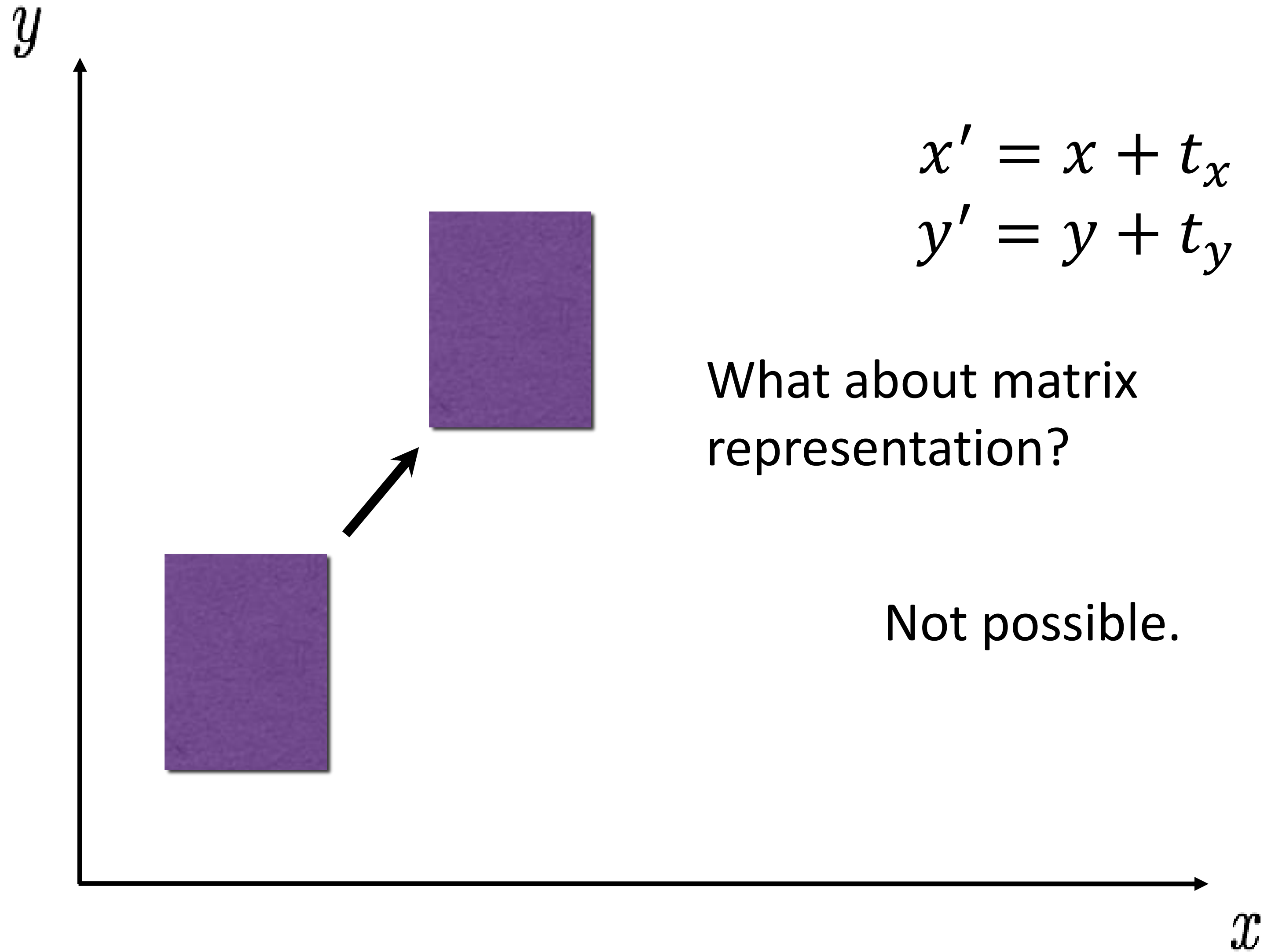


$$x' = x + t_x$$
$$y' = y + t_y$$

What about matrix representation?

$$\mathbf{M} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Translation



$$x' = x + t_x$$
$$y' = y + t_y$$

What about matrix representation?

Not possible.

Homogeneous coordinates

- **Homogeneous coordinates** represent 2D point with a 3D vector.

heterogeneous coordinates		homogeneous coordinates
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\rightarrow$	$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Homogeneous coordinates

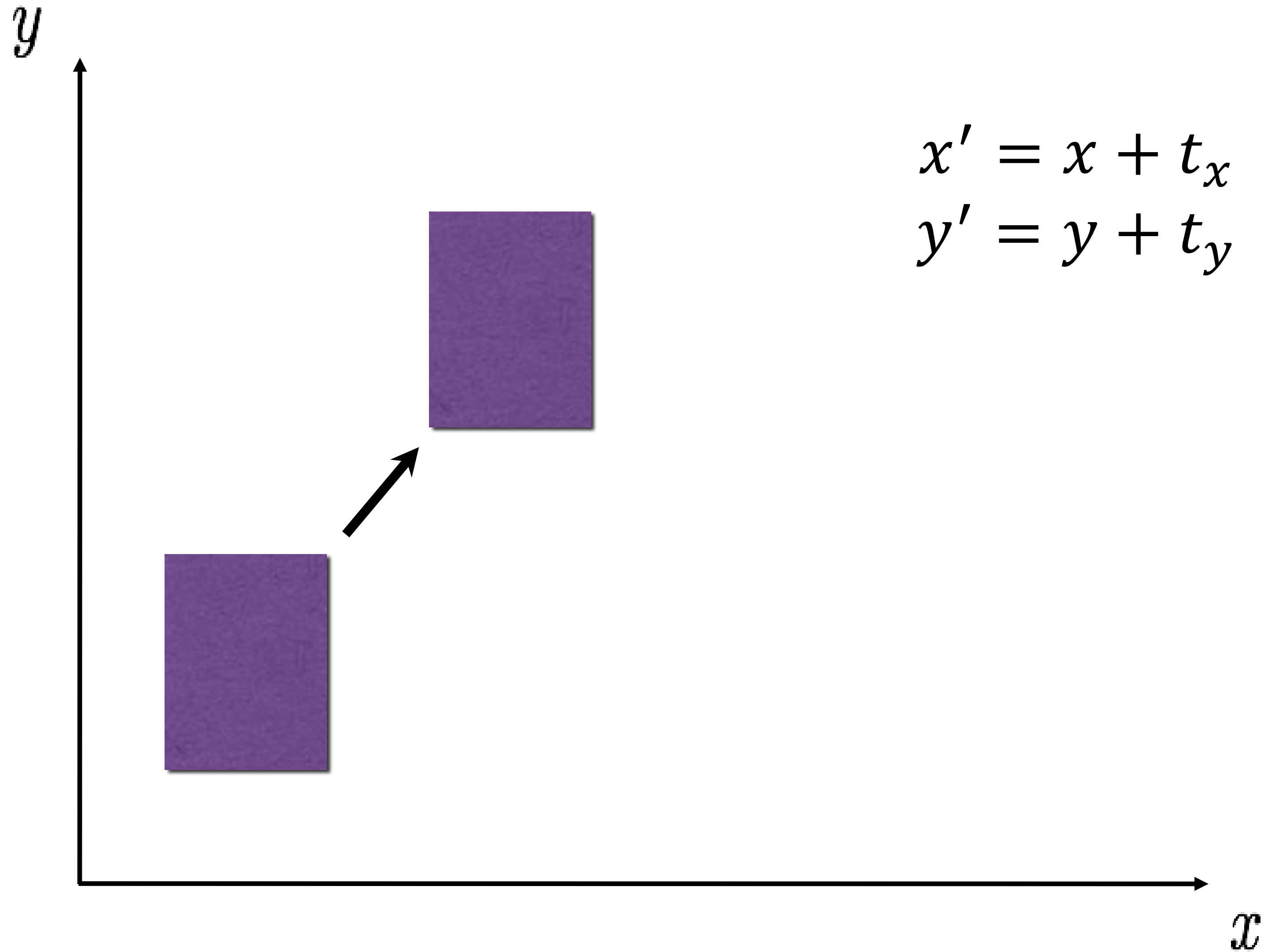
- **Homogeneous coordinates** represent 2D point with a 3D vector.
- Homogeneous coordinates are only defined up to scale.

heterogeneous coordinates | homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

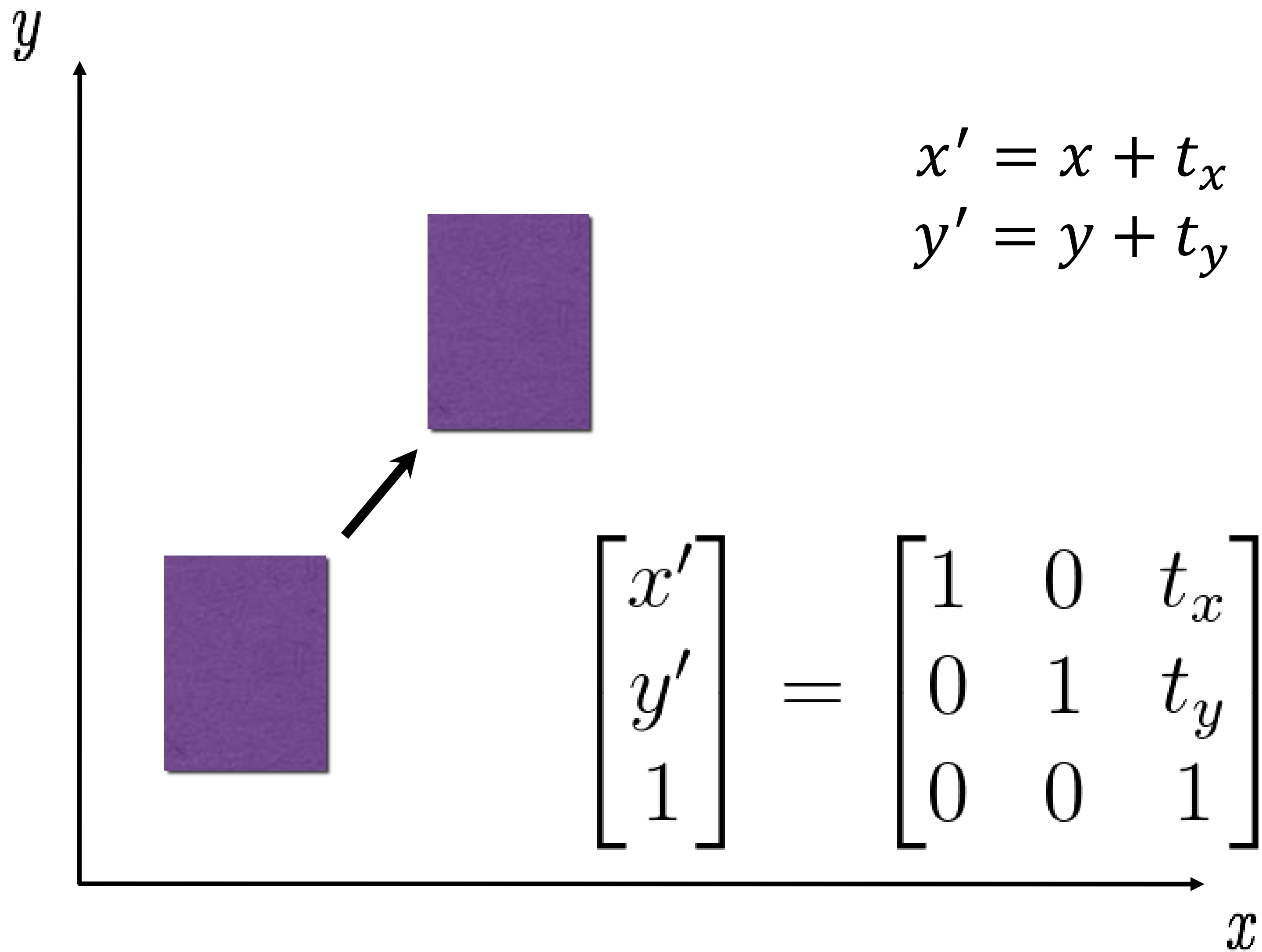
Translation

- How do we do it now?



$$x' = x + t_x$$
$$y' = y + t_y$$

Translation



$$x' = x + t_x$$
$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Side note: linear transformation

- Linear transformation are Transformation that meets additively and scalar multiplication conditions:

$$f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$$
$$f(c\boldsymbol{u}) = cf(\boldsymbol{u})$$

- Translation is **not** a linear transformation since it doesn’t meet the scalar multiplication condition.
- Properties of linear transformations:
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved

Affine Transformations

- Affine transformations are combinations of linear transformations and translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \textbf{ or } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

- Properties of affine transformations:
  - ~~Origin maps to origin~~
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved

Affine transformation: example

- Translate then scale vs. scale then translate :

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & at_x \\ 0 & b & bt_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\neq$$

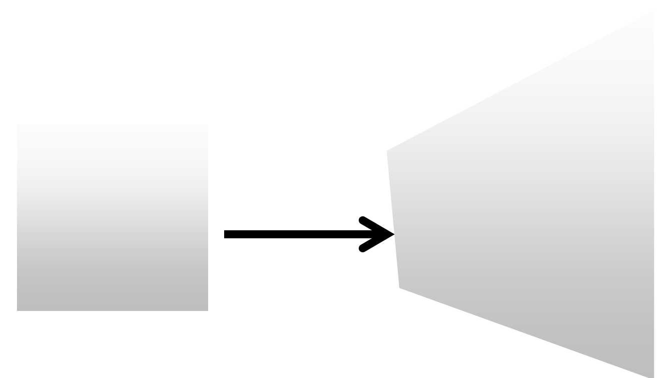
$$\begin{bmatrix} x'' \\ y'' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & t_x \\ 0 & b & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Order of matrices **DOES** matter ($A \cdot B \neq B \cdot A$)

Projective transformation

- Also known as **homography** or **homographic transformation**.
- A generalization of affine transformation.
- Properties of projective transformations:
  - ~~Origin maps to origin~~
  - Lines map to lines
  - ~~Parallel lines remain parallel~~
  - ~~Ratios are preserved~~

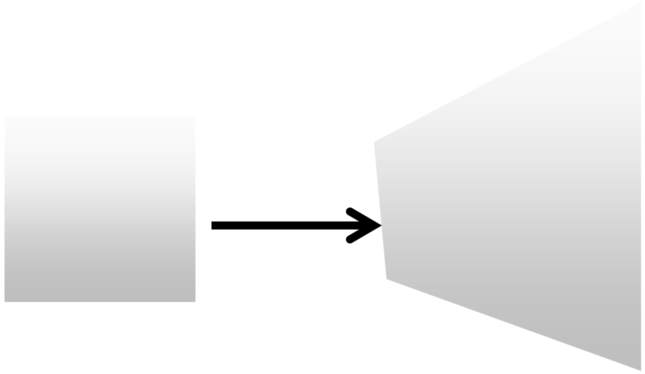
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$



Projective transformation

- How many DOFs do we have here?

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$



Projective transformation

- How many DOFs do we have here?
  - We have 9 variables to the projection matrix.
  - Because we use homogenous vectors we can say:

$$\alpha \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  - We can add a constraint like $\sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2} = 1$ and still get a valid answer.
  - 9 variables – 1 constraint = 8 DOF

contents

- 2D->2D transformations
- **3D->3D transformations**
- 3D->2D transformations (3D projections)
  - Perspective projection
  - Orthographic projection

3D->3D transformations

- Exactly the same as 2D->2D transformations from earlier, just with 3D.
- What do we see here?

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & cos\theta & -sin\theta & t_y \\ 0 & sin\theta & cos\theta & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

3D->3D transformations

- Exactly the same as 2D->2D transformations from earlier, just with 3D.
- What do we see here?

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & cos\theta & -sin\theta & t_y \\ 0 & sin\theta & cos\theta & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

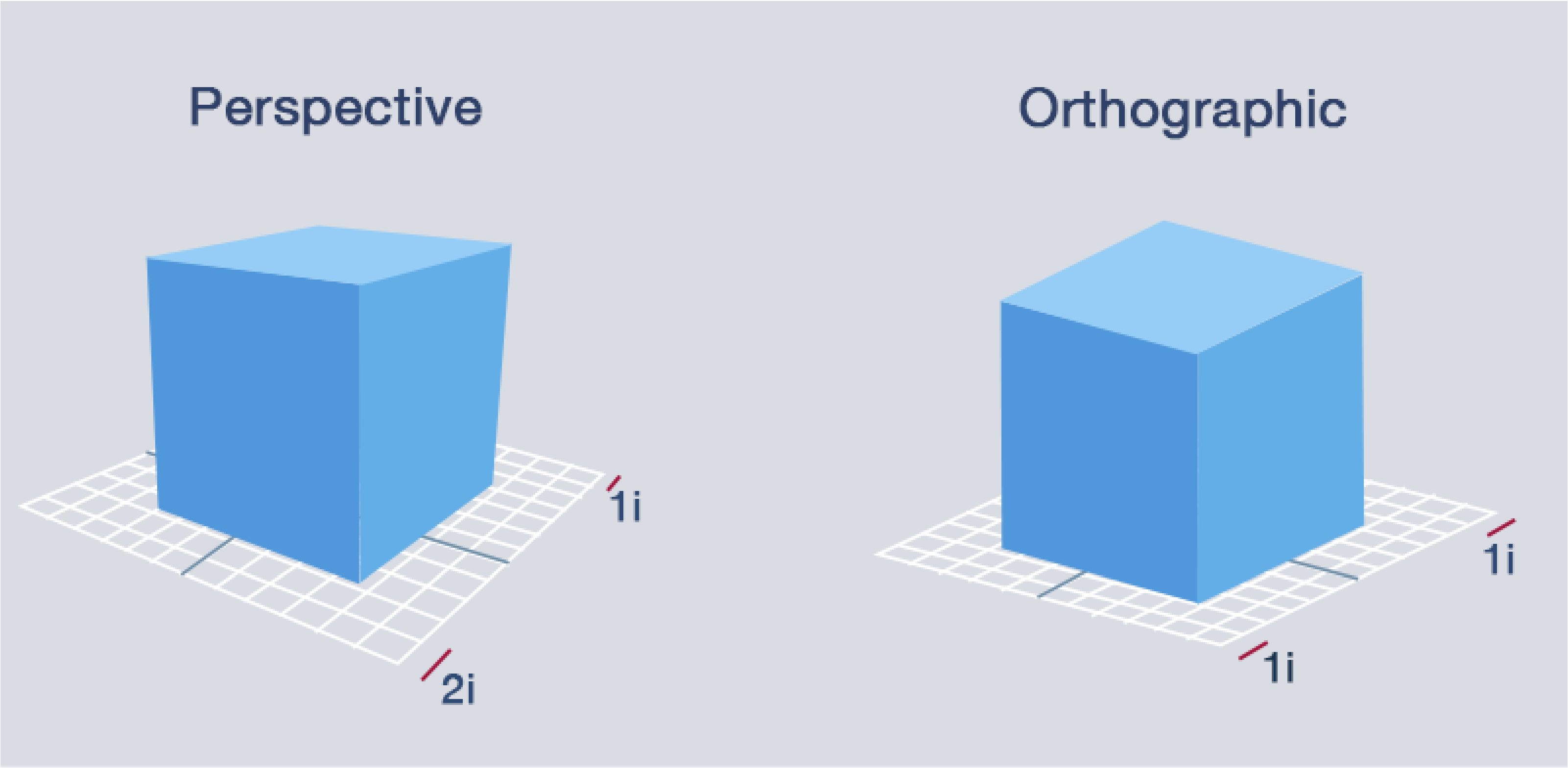
  – Rotation around x axis and then translation

contents

- 2D->2D transformations
- 3D->3D transformations
- 3D->2D transformations (3D projections)
  - **Perspective projection**
  - Orthographic projection

3D projection

- **3D projection** is any method of mapping three-dimensional points to a two-dimensional plane.
- Two types of projections are **orthographic** and **perspective**.

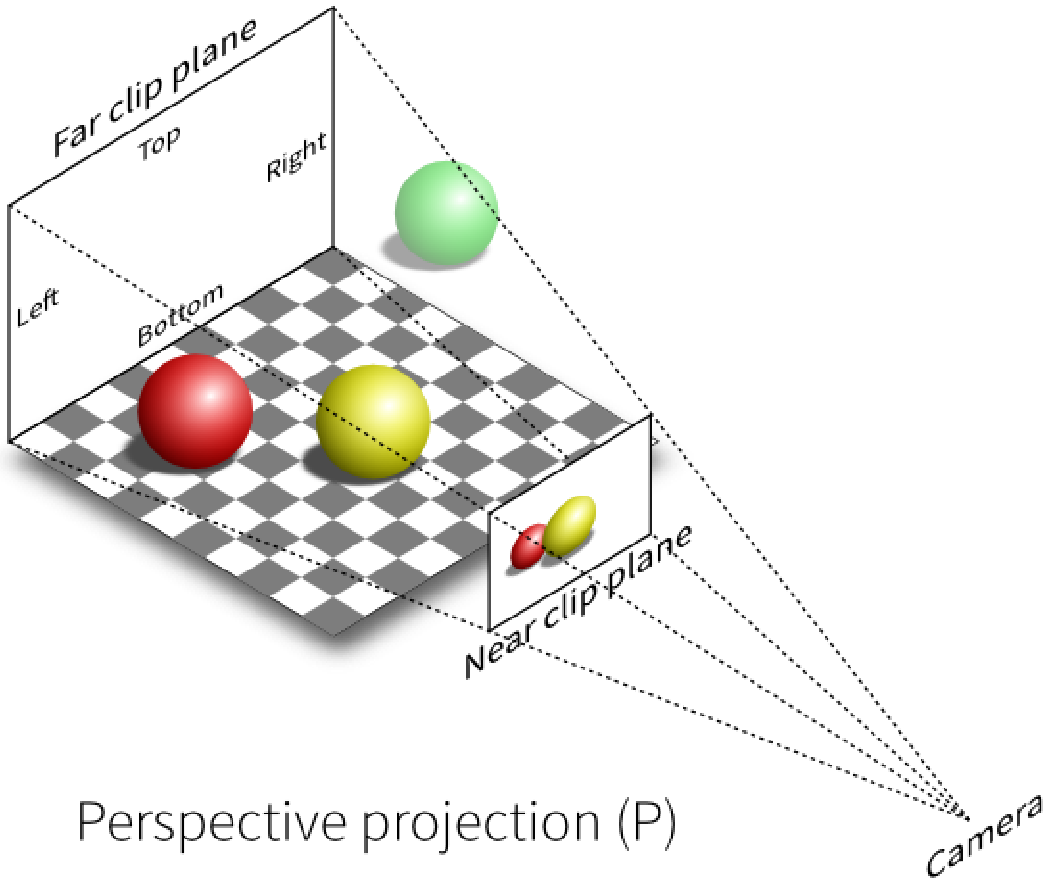


Perspective- definition

1. the art of drawing solid objects on a two-dimensional surface so as to give the right impression of their height, width, depth, and position in relation to each other when viewed from a particular point.
2. a particular attitude toward or way of regarding something; a point of view.

Perspective projection

- Perspective projection is the kind of projection we get from a regular image of a regular (pinhole) camera.



Perspective projection (P)

perspective manipulation

Street art- perspective manipulation

perspective manipulation- Ames Room

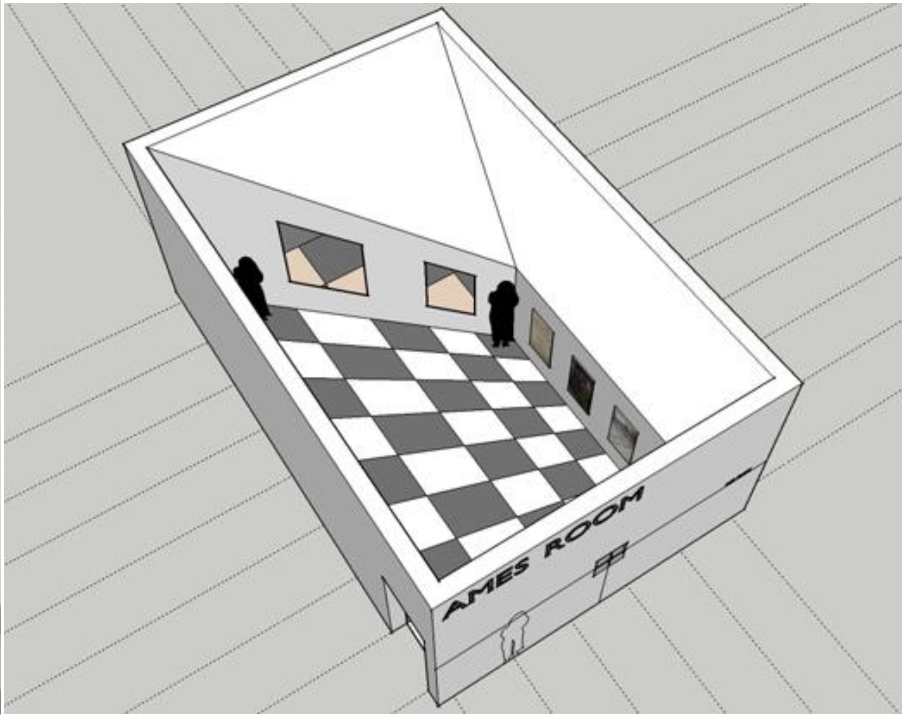
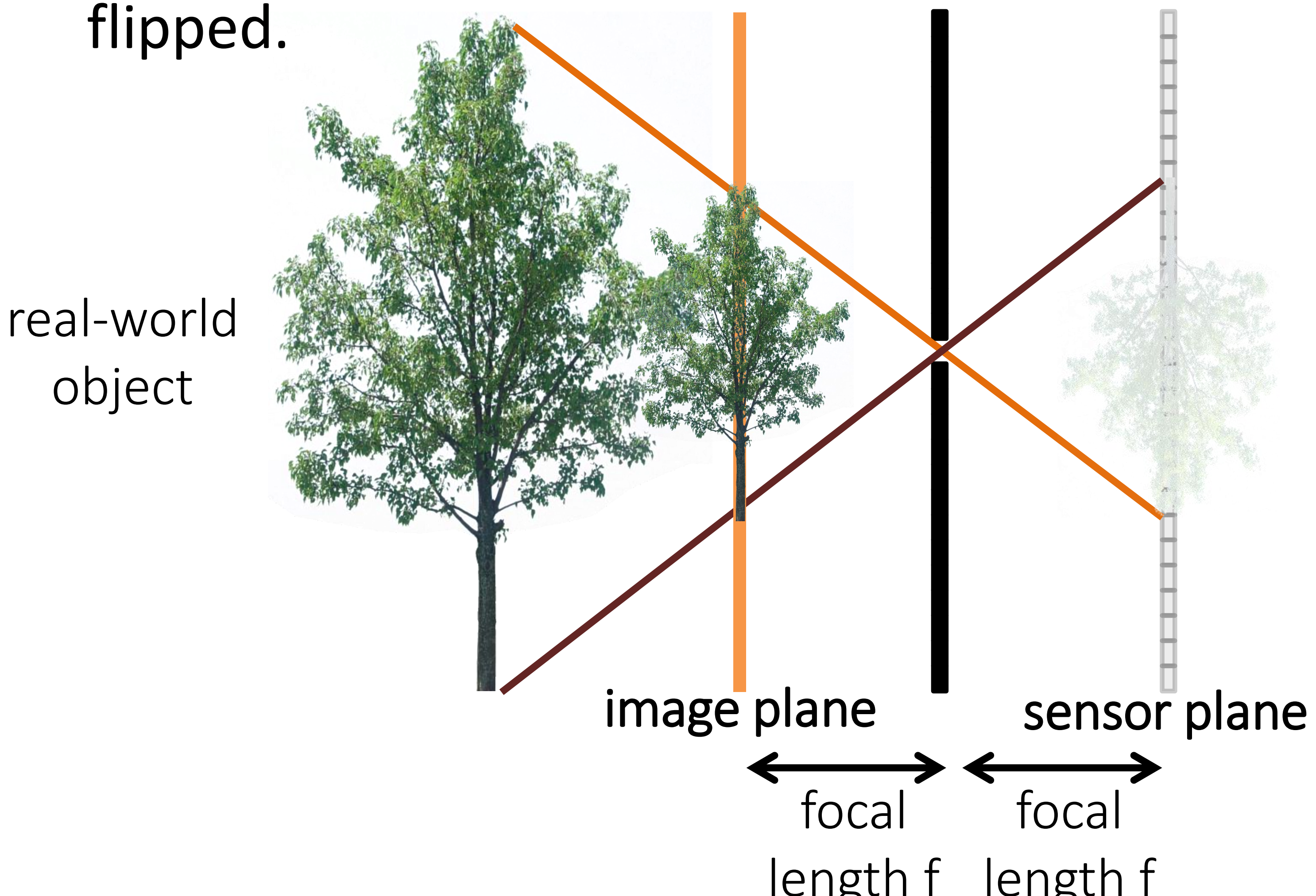


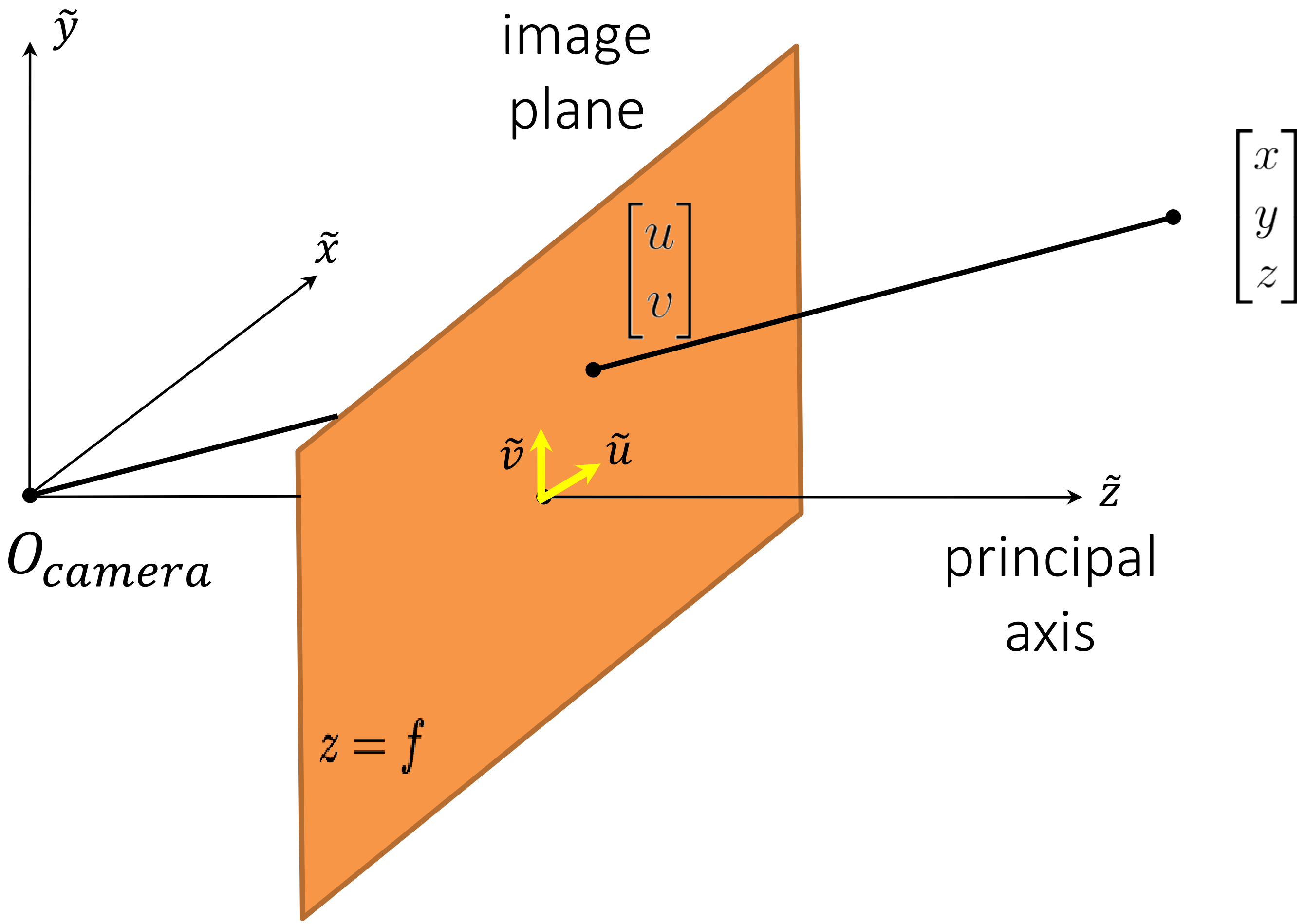
Image plane

- When dealing with imaged 3D scenes, we tend to use the **image plane** rather than the sensor plane which is flipped.



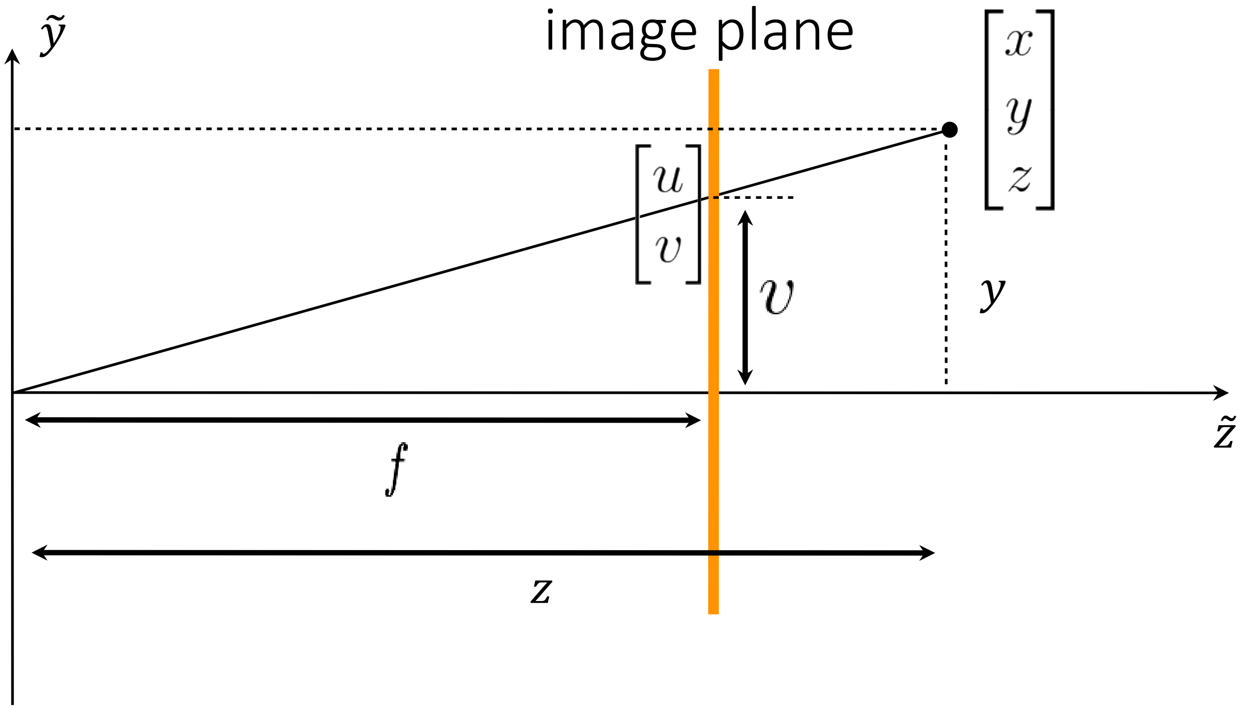
Perspective projection

- **Perspective projection** (also known as **perspective transformation**) is a linear projection where three dimensional objects are projected on the image plane.



Perspective projection

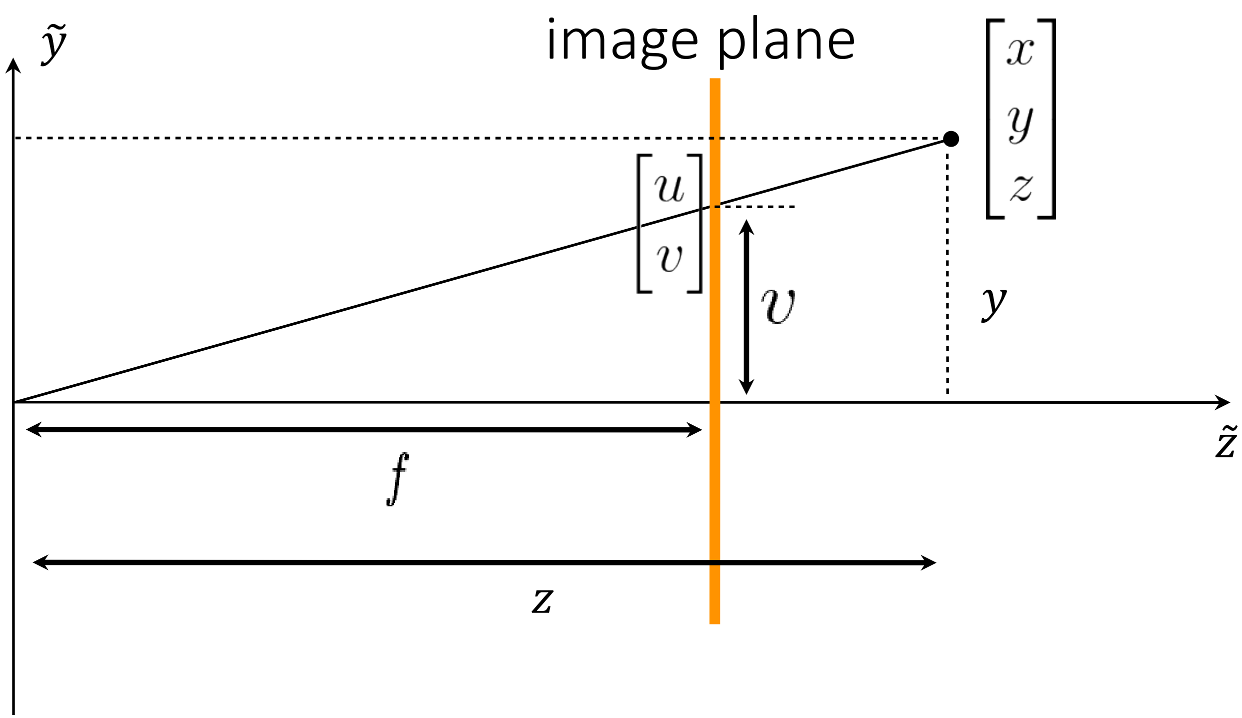
- What is the relationship between y & v?



Perspective projection

- Using triangle proportions (Thales' theorem) we can easily conclude that:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f\frac{x}{z} \\ f\frac{y}{z} \end{bmatrix}$$



Perspective projection

- Let's use the homogeneous coordinates:

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix} \mapsto \begin{bmatrix} f\frac{x}{z} \\ f\frac{y}{z} \end{bmatrix}$$

  – Units of $[m]$

Perspective projection

- Let's split into 2 matrices and use 3D->2D homogenous coordinates:

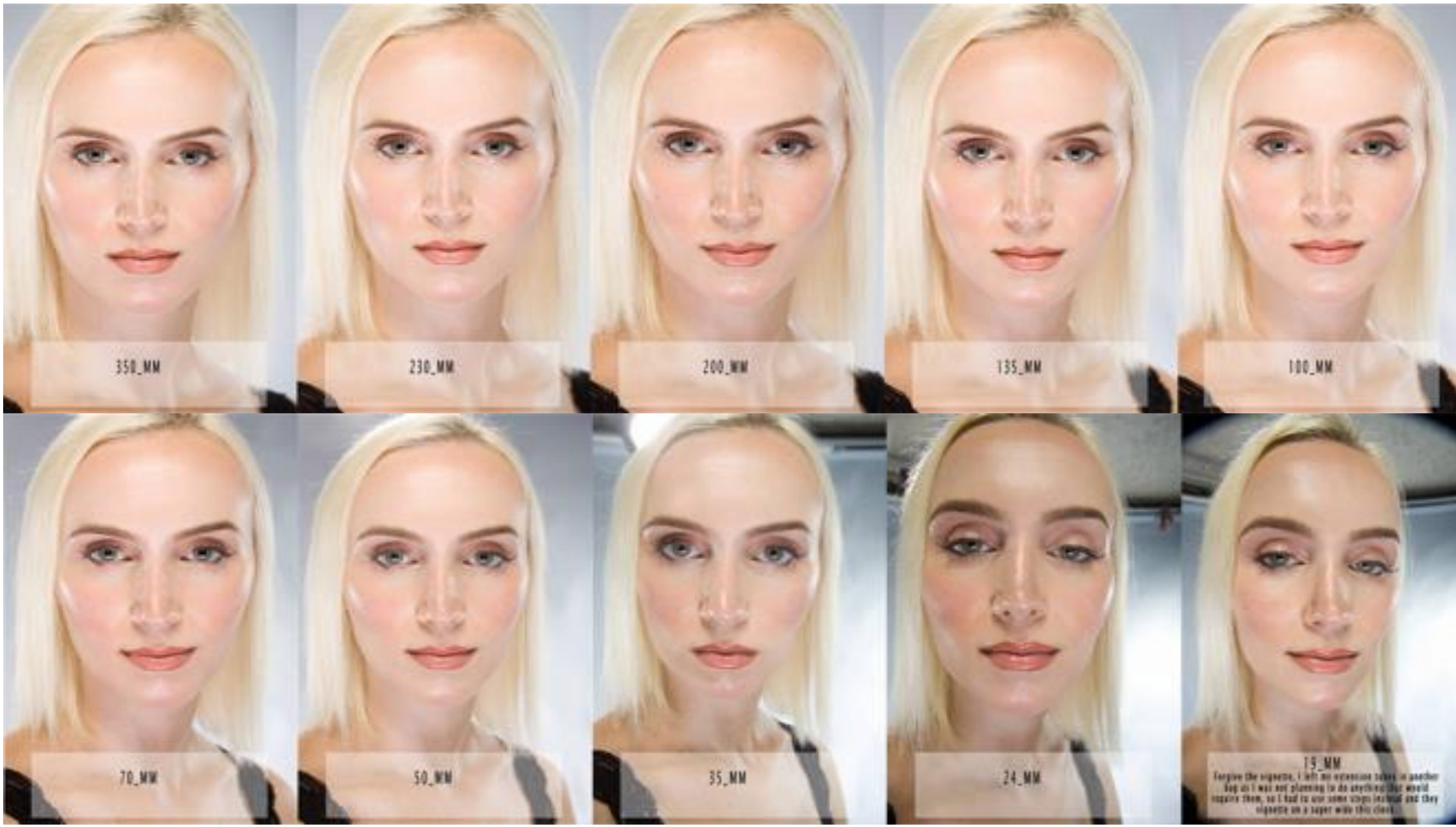
$$\underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Intrinsic camera matrix}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Projection matrix}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix} \mapsto \begin{bmatrix} f\frac{x}{z} \\ f\frac{y}{z} \end{bmatrix}$$

Vertigo effect

- Has several different names (**vertigo effect, dolly zoom, lens compression, perspective distortion**) that all mean the same thing.
- intro:
  - https://www.youtube.com/watch?v=UrhtKvBMZ3g (until 01:50)

Vertigo effect

https://www.youtube.com/watch?v=_TTXY1Se0eg(until 02:55)



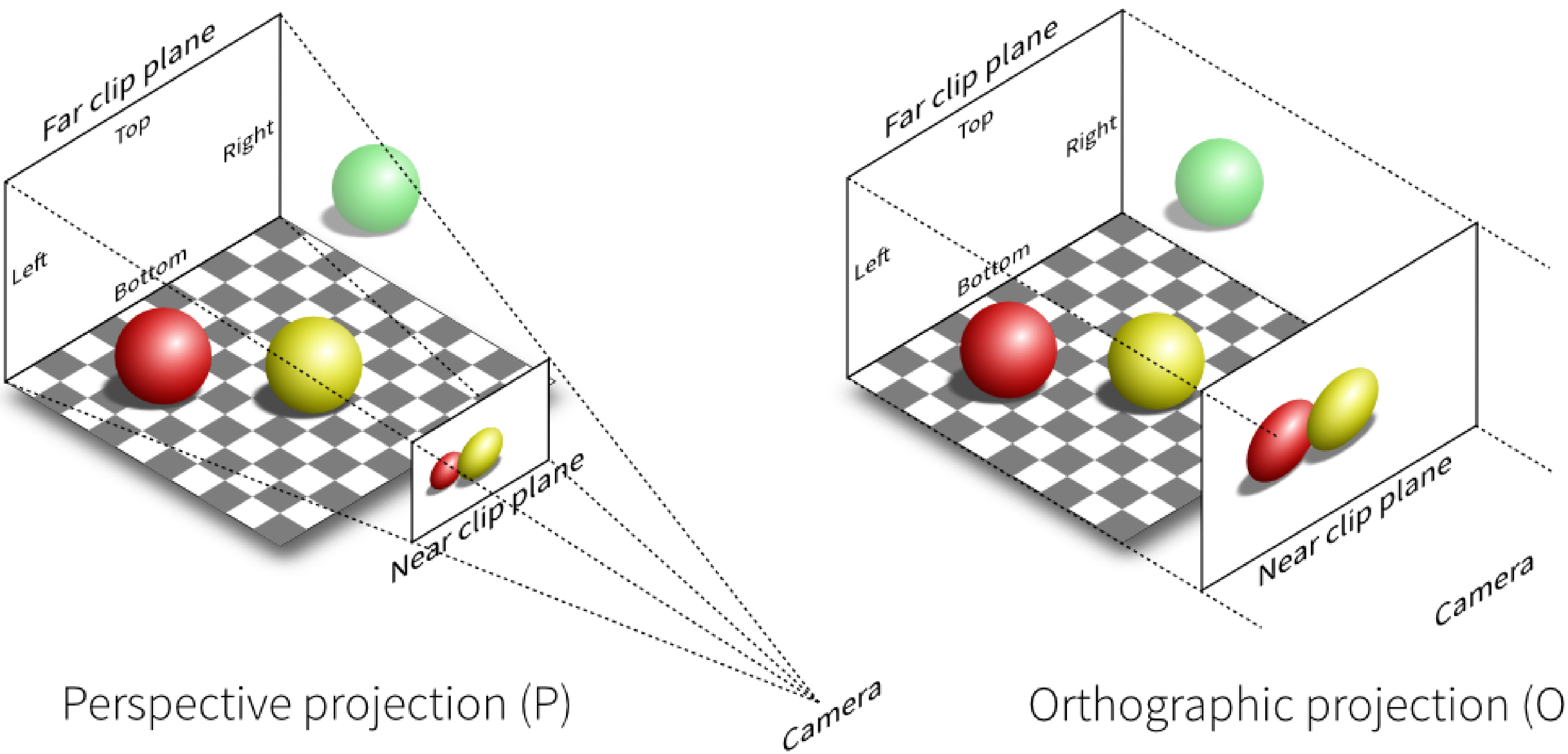
Vertigo effect

contents

- 2D->2D transformations
- 3D->3D transformations
- 3D->2D transformations (3D projections)
  - Perspective projection
  - **Orthographic projection**

Orthographic projection

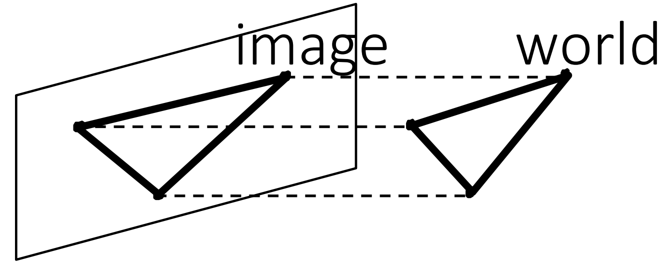
- A different kind of camera model that can be used is **orthographic projection** or **orthographic camera**.
- This kind of projection is invariant to the distance from the camera, and only depends on the object's size.



Perspective projection (P)

Orthographic projection (O)

Orthographic projection



- Orthographic matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

- Weak perspective matrix (with scale coefficient)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z_0 \end{bmatrix} \mapsto \begin{bmatrix} x/z_0 \\ y/z_0 \end{bmatrix}$$

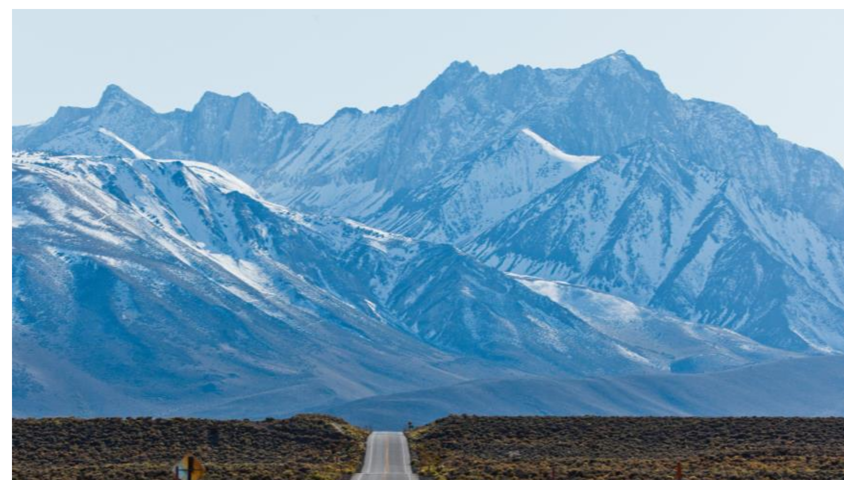
Orthographic projection

- When can we assume a weak perspective camera?

Orthographic projection

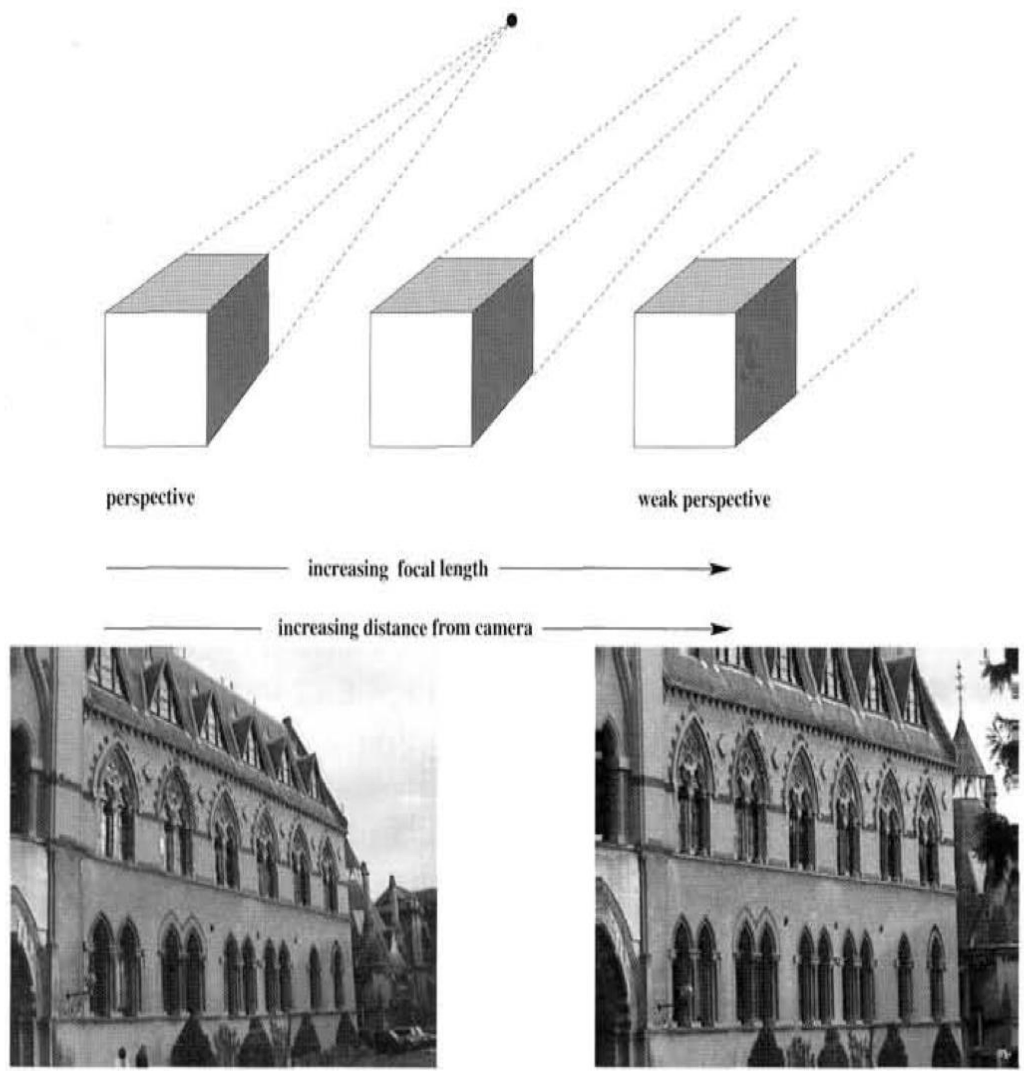
- When can we assume a weak perspective camera?
- When dealing with a plane parallel to image plane- z_0 is the distance to this plane.

- When far away objects- we can assume the average distance to the objects as z_0.



Weak perspective camera

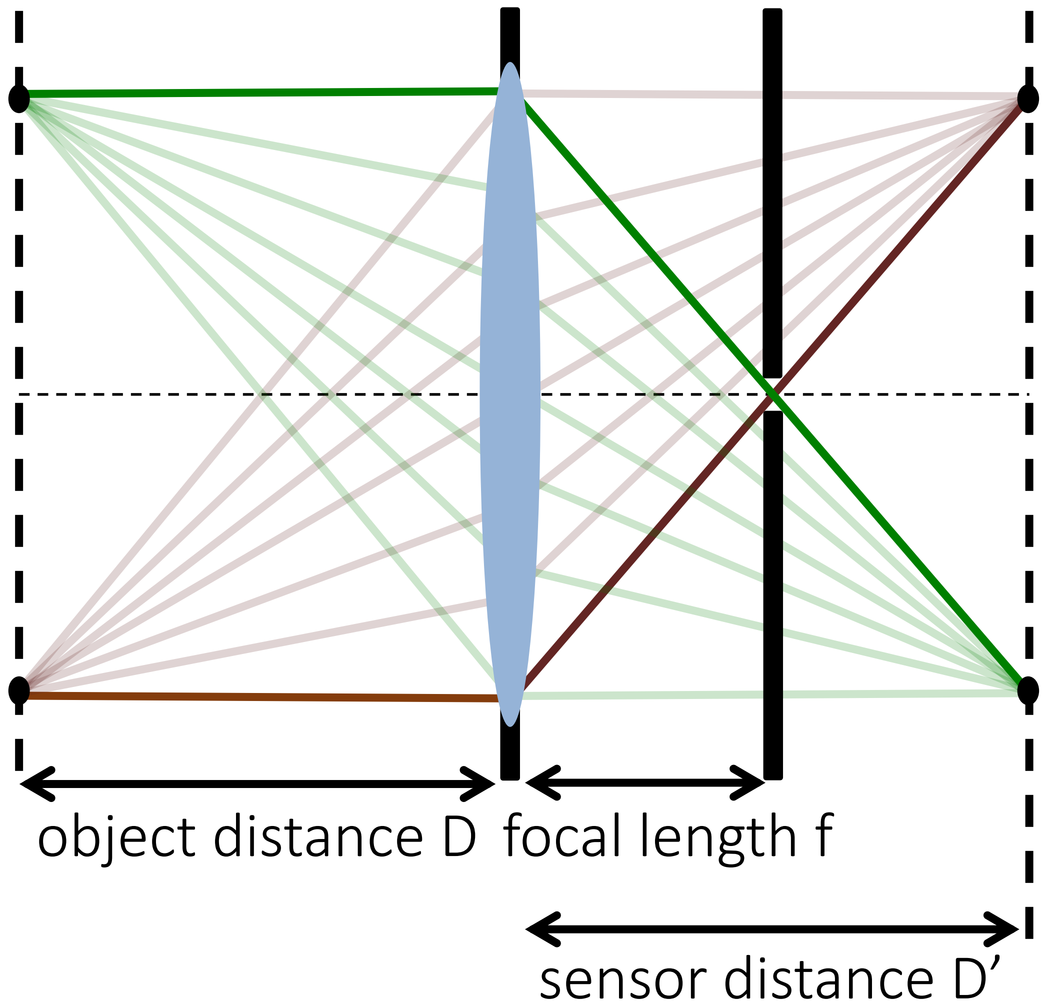
- One way to transform a regular perspective image to an orthographic view is simply taking the picture from a distance with zoom (large focal length).



Weak perspective camera

- Real orthographic camera:

Place a pinhole at focal length, so that only rays parallel to primary ray pass through.



set the sensor distance as
$$D' = 2f$$
in order to achieve unit magnification.

Weak perspective camera

perspective camera

weak perspective camera

Orthographic projection

- Why we want to assume a weak perspective camera?

Orthographic projection

- Why we want to assume a weak perspective camera?
- Easier to do a lot of image manipulation. For example: image stitching (no projective transformation, just affine), called panograma.